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Quadratic and cubic transformations and zeros of hypergeometric polynomials

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Abstract

In this paper we consider the location of the zeros of the hypergeometric polynomials that lie in either the quadratic or the cubic class, where each of these classes is determined by a necessary and sufficient condition due to Kummer. We show that the zeros of most polynomials in these classes can be specified by simple applications of the results proved in recent papers of Driver and Duren. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

The Gauss hypergeometric function $F(a,b;c;z)$ is defined by

$$F(a,b;c;z) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \quad |z| < 1,$$

where a, b and c are complex parameters and

$$(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1) = \Gamma(\alpha+k)/\Gamma(\alpha)$$

is Pochhammer's symbol. When $a = -n$ is a negative integer, the series terminates and reduces to a polynomial of degree at most n, called a hypergeometric polynomial.

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We are interested in the location of the zeros of $F(-n,b;c;z)$ for all possible values of the free parameters b and c. Throughout this paper we shall consider only real values of b and c. It follows immediately from the definition of the Pochhammer symbol that if $b=-m$ where $m<n$, $m\in\mathbb{N}$, $F(-n,b;c;z)$ reduces to a polynomial of degree m. However, since we are interested in the location of the zeros of F as b and/or c vary, we shall consider $F(-n,-m;c;z)$, $m\in\mathbb{N}$, $m<n$ as $\lim_{b\to -m}F(-n,b;c;z)$. Also, we note that $F(-n,b;c;z)$ is not defined when $c=0,-1,\ldots,-n+1$.

In general, very little is known about the behaviour of the zeros of hypergeometric polynomials but complete results on the location of the zeros have been obtained for the special case $F(-n,b;2b;z)$ where b is real (cf. [4,5]). In this paper we focus our attention on the location of the zeros of all hypergeometric polynomials that lie in either the quadratic or the cubic class, where each of these classes is determined by a necessary and sufficient condition due to Kummer (cf. [3], pp. 64, 67).

It is well known that the Gegenbauer or ultraspherical polynomials, defined by the generating relation

$$(1-2zr+r^2)^{-\lambda}=\sum_{n=0}^{\infty}C_n^{\lambda}(z)r^n,$$

are orthogonal on the interval $(-1,1)$ with respect to the weight function $(1-x^2)^{\lambda-1/2}$ for $\lambda>-\frac{1}{2}$. Connections between the hypergeometric polynomials $F(-n,b;c;z)$ and $C_n^{\lambda}(z)$ have been used in the proofs of the following two theorems which will be invoked in the proofs of our results.

Theorem 1.1 (cf. Driver and Duren [4,5]). *Let $F=F(-n,b;2b;z)$, where b is real.*

(i) *For $b>-\frac{1}{2}$, all zeros of F are simple and lie on the circle $|z-1|=1$.*
(ii) *For $-\frac{1}{2}-j<b<\frac{1}{2}-j$, $j=1,2,\ldots,[n/2]-1$, $(n-2j)$ zeros of F lie on the circle $|z-1|=1$. If $j=2k$ is even, there are k non-real zeros of F in each of the four regions bounded by the circle $|z-1|=1$ and the real axis. If $j=2k+1$ is odd, there are k non-real zeros of F in each of the four regions described above and the remaining two zeros are real.*
(iii) *If n is even, for $-[n/2]<b<-[n/2]+\frac{1}{2}$, no zeros of F lie on $|z-1|=1$. If $n=4k$, all zeros of F are non-real whereas if $n=4k+2$, two zeros of F are real and $4k$ are non-real. If n is odd, for $-1-[n/2]<b<-[n/2]+\frac{1}{2}$, only the fixed real zero of F at $z=2$ lies on $|z-1|=1$. If $n=4k+1$, $n-1=4k$ zeros of F are non-real whereas if $n=4k+3$, two further zeros are real and the remaining $4k$ are non-real*
(iv) *For $j-n<b<j-n+1$, $j=1,2,\ldots,[n/2]-1$, $(n-2j)$ zeros of F are real and greater than 1. If $j=2k$ is even, all remaining $2j$ zeros of F are non-real with k zeros in each of the regions described above; while if $j=2k+1$, $4k$ zeros are non-real as before and 2 are real.*
(v) *For $b<1-n$, all zeros of F are real and greater than 1. As $b\to-\infty$, all the zeros of F converge to the point $z=2$.*

Theorem 1.2 (cf. Driver and Duren [6]). *Let $C_n^{\lambda}(z)$ be the Gegenbauer polynomial with real parameter λ.*

(i) *For $\lambda>-\frac{1}{2}$, all zeros of $C_n^{\lambda}(z)$ are real and simple and lie in $(-1,1)$.*
(ii) *For $-\frac{1}{2}-j<\lambda<\frac{1}{2}-j$, $j=1,2,\ldots,[n/2]-1$, $(n-2j)$ zeros of $C_n^{\lambda}(z)$ lie in $(-1,1)$. The remaining $2j$ zeros lie outside $(-1,1)$ with exactly j zeros in each of the left and right half*

planes. If $j=2k$ is even, then exactly k zeros are in each quadrant of the complex plane and they are all non-real. If $j=2k+1$ is odd, k zeros lie in each of the four quadrants while the remaining two are on the real axis, one in the interval $(1,\infty)$, the other one in $(-\infty,-1)$.

(iii) *If n is even, for $-[n/2] < \lambda < -[n/2]+\frac{1}{2}$, no zeros of $C_n^{\lambda}(z)$ lie in $(-1,1)$. If $n=4k$, all zeros of $C_n^{\lambda}(z)$ are non-real with exactly k in each quadrant. If $n=4k+2$, two zeros of $C_n^{\lambda}(z)$ are real and $4k$ are non-real with k in each quadrant. If n is odd, for $-1-[n/2] < \lambda < -[n/2]+\frac{1}{2}$, only the fixed real zero of $C_n^{\lambda}(z)$ at the origin lies in $(-1,1)$. If $n=4k+1$, $n-1=4k$ zeros of $C_n^{\lambda}(z)$ are non-real with k in each quadrant whereas if $n=4k+3$, $C_n^{\lambda}(z)$ has $4k$ non-real zeros with k in each quadrant, and two additional real zeros, one in $(1,\infty)$ and one in $(-\infty,-1)$.*

(iv) *For $j-n < \lambda < j-n+1$, $j=1,2,\ldots,[n/2]-1$, $(n-2j)$ zeros of $C_n^{\lambda}(z)$ lie on the imaginary axis symmetrically with respect to the real axis. For the $2j$ remaining zeros, if $j=2k$ is even, then k zeros are situated in each quadrant and none are on the real axis. If $j=2k+1$ is odd, then k of the zeros are in each quadrant and the two remaining zeros are on the real axis, one in $(1,\infty)$, the other in $(-\infty,-1)$.*

(v) *For $\lambda < 1-n$, all zeros of $C_n^{\lambda}(z)$ lie on the imaginary axis and tend symmetrically to the origin as $\lambda \to -\infty$.*

2. The quadratic class of hypergeometric polynomials

A necessary and sufficient condition for a hypergeometric function to admit a quadratic transformation (cf. [10] Section 5.8 or [1], p. 560) is that the numbers $\pm(1-c), \pm(a-b), \pm(a+b-c)$ are such that one of them is equal to $\frac{1}{2}$ or two of them are equal. This condition generates a class of twelve functions and a complete list is given in [10], p. 124. If we fix $a=-n$, the hypergeometric polynomials in the quadratic class are

$$F\left(-n,b;\frac{1}{2};z\right),\quad F\left(-n,b;-n+b+\frac{1}{2};z\right),\quad F\left(-n,-n+\frac{1}{2};c;z\right),$$

$$F\left(-n,b;\frac{3}{2};z\right),\quad F\left(-n,b;-n+b-\frac{1}{2};z\right),\quad F\left(-n,-n-\frac{1}{2};c;z\right),$$

$$F(-n,b;2b;z),\quad F(-n,b;-n-b+1;z),\quad F\left(-n,b;\frac{-n+b+1}{2};z\right),$$

$$F(-n,b;-2n;z),\quad F(-n,b;b+n+1;z),\quad F(-n,1+n;c;z).$$

The two identities

$$F(-n,b;c;1-z)=\frac{(c-b)_n}{(c)_n}F(-n,b;1-n+b-c;z), \tag{2.1}$$

$$F(-n,b;c;z)=\frac{(b)_n}{(c)_n}(-z)^nF\left(-n,1-c-n;1-b-n;\frac{1}{z}\right), \tag{2.2}$$

where b and c are real and $c\notin\{-n+1,-n+2,\ldots,0\}$, link the polynomials across each row of the above list and therefore the problem of locating the zeros of any polynomial in the quadratic class reduces to finding the zeros of the polynomials in the first column. The identity (2.1) is Pfaff's formula (cf. [2], (2.3.14)), and the identity (2.2) follows directly from the definition of F along with standard manipulation of the Pochhammer symbol.

Considering each of the four polynomials in the first column above, we note first that the zeros of $F(-n,b;2b;z)$ are described in Theorem 1.1. Both $F(-n,b;\frac{1}{2};z)$ and $F(-n,b;\frac{3}{2};z)$ can be expressed in terms of Gegenbauer polynomials, namely (cf. [8,9], Eqs. (162), (163))

$$F\left(-n,b;\frac{1}{2};z\right)=\frac{n!}{(1-b)_n}C_{2n}^{b-n}(\sqrt{z}),\quad b\neq 1,2,\ldots,n, \tag{2.3}$$

$$F\left(-n,b;\frac{3}{2};z\right)=-\frac{n!}{2(1-b)_{n+1}}\frac{1}{\sqrt{z}}C_{2n+1}^{b-n-1}(\sqrt{z}),\quad b\neq 1,2,\ldots,n+1. \tag{2.4}$$

The fourth polynomial $F(-n,b;-2n;z)$ cannot be expressed in terms of a Gegenbauer polynomial and this interesting case is the subject of a separate paper (cf. [7]), where results concerning numbers of real zeros and also the asymptotic zero distribution as $n\to\infty$ can be found.

Remark 2.1. In (2.3), we observe that $(1-b)_n=0$ for $b=1,2,\ldots,n$, whereas $F(-n,b;\frac{1}{2};z)$ is well defined for these values of b. This apparent anomaly is explained if we examine the generating relation for $C_n^{\lambda}(z)$, namely

$$(1-2zr+r^2)^{-\lambda}=\sum_{n=0}^{\infty}C_n^{\lambda}(z)r^n.$$

One obtains explicitly

$$C_n^{\lambda}(z)=\sum_{k=[n+1/2]}^{n}\frac{(\lambda)_k}{(n-k)!(2k-n)!}2^{2k-n}(-1)^{n-k}z^{2k-n},$$

which is a polynomial of degree n with leading coefficient $(\lambda)_n 2^n/n!$ Since $(\lambda)_k=0$ for $\lambda=0,-1,\ldots,-k+1$, we see that $C_n^{\lambda}(z)$ is a polynomial of degree $2m-n$ for $\lambda=-m$, $m=[n+1/2],\ldots,n-1$; whereas $C_n^{\lambda}(z)$ is identically zero for $\lambda=-m$, $m=0,\ldots,[n-1/2]$. It follows that $C_{2n}^{b-n}(\sqrt{z})$ is identically zero for $b=1,2,\ldots,n$ and (2.3) makes sense for these values of b provided one considers appropriate limits.

Theorem 2.2. *Let $F=F(-n,b;\frac{1}{2};z)$ with b real.*

(i) *For $b>n-\frac{1}{2}$, all n zeros of F are real and simple and lie in $(0,1)$.*
(ii) *For $n-\frac{1}{2}-j<b<n+\frac{1}{2}-j$, $j=1,2,\ldots,n-1$, $(n-j)$ zeros of F lie in $(0,1)$ and the remaining j zeros of F form $[j/2]$ non-real complex pairs of zeros and one real zero lying in $(1,\infty)$ when j is odd.*
(iii) *For $0<b<\frac{1}{2}$, F has $[n/2]$ non-real complex conjugate pairs of zeros with one real zero in $(1,\infty)$ when n is odd.*
(iv) *For $-j<b<-j+1$, $j=1,2,\ldots,n-1$, F has exactly j real negative zeros. There is exactly one further real zero greater than 1 only when $(n-j)$ is odd and all the remaining zeros of F are non-real.*
(v) *For $b<1-n$, all zeros of F are real and negative and converge to zero as $b\to-\infty$.*

Proof. Eq. (2.3) together with Theorem 1.2 provides the proof if we apply simple properties of the transformation $w=\sqrt{z}$ and observe that C_{2n}^{λ} is an even function. $\square$

Theorem 2.3. *Let $F=F(-n,b;\frac{3}{2};z)$ with b real.*

(i) *For $b>n+\frac{1}{2}$, all n zeros of F are real and simple and lie in $(0,1)$.*
(ii) *For $n+\frac{1}{2}-j<b<n+\frac{3}{2}-j$, $j=1,2,\ldots,n-1$, $(n-j)$ zeros of F lie in $(0,1)$. F has $[j/2]$ pairs of non-real complex conjugate zeros with one additional real zero in $(1,\infty)$ when j is odd.*
(iii) *For $0<b<\frac{3}{2}$, F has $[n/2]$ pairs of non-real complex conjugate zeros and one zero in $(1,\infty)$ when n is odd.*
(iv) *For $-j<b<-j+1$, $j=1,2,\ldots,n-1$, F has exactly j real negative zeros. There is exactly one further real zero in $(1,\infty)$ only when $(n-j)$ is odd and all remaining zeros of F are non-real.*
(v) *For $b<1-n$, all zeros of F are real and negative and converge to zero as $b\to-\infty$.*

Proof. Eq. (2.4) together with Theorem 1.2 furnishes the proof, noting that in (2.4), the simple zero of $C_{2n+1}^{\lambda}(\sqrt{z})$ at the origin is cancelled out by the factor $1/\sqrt{z}$. $\square$

3. Cubic transformations

The hypergeometric polynomial $F(-n,b;c;z)$ has a cubic transformation (cf. [3], p. 67) if either two of the numbers

$$\pm(1-c),\quad \pm(-n-b),\quad \pm(c+n-b) \tag{3.1}$$

are equal to one-third, or

$$1-c=\pm(-n-b)=\pm(c+n-b). \tag{3.2}$$

Conditions (3.1) and (3.2) give rise to twelve and four hypergeometric polynomials, respectively, in which n is the only free parameter. A complete list is

$$F\left(-n,-n-\frac{1}{3};-2n;z\right),\quad F\left(-n,-n-\frac{1}{3};\frac{2}{3};z\right),\quad F\left(-n,n+1;\frac{4}{3};z\right),$$
$$F\left(-n,-n+\frac{1}{3};-2n;z\right),\quad F\left(-n,-n+\frac{1}{3};\frac{4}{3};z\right),\quad F\left(-n,n+1;\frac{2}{3};z\right),$$
$$F\left(-n,-n-\frac{1}{3};-2n-\frac{2}{3};z\right),\quad F\left(-n,-n-\frac{1}{3};\frac{4}{3};z\right),\quad F\left(-n,n+\frac{5}{3};\frac{4}{3};z\right),$$
$$F\left(-n,-n+\frac{1}{3};-2n+\frac{2}{3};z\right),\quad F\left(-n,-n+\frac{1}{3};\frac{2}{3};z\right),\quad F\left(-n,n+\frac{1}{3};\frac{2}{3};z\right),$$
$$F(-n,n+1;2n+2;z),\quad F(-n,n+1;-2n;z),\quad F(-n,-3n-1;-2n;z),$$
$$F\left(-n,\frac{-n+1}{3};\frac{-2n+2}{3};z\right).$$

The list is given in row format because, once again, the identities (2.1) and (2.2) link the polynomials in each row. We will show that the twelve polynomials in the first four rows of the above list, which arise from condition (3.1), all have n real zeros lying in specified intervals. The remaining four will be discussed separately. We shall need the following theorem (cf. [7], Theorem 3.1 and Corollary 3.2).

Theorem 3.1. *If $b \in (-n-1, -n)$ or if $b \in (-n, -n+1)$, all n zeros of $F(-n, b; -2n; z)$ are real and lie in the interval $(1, \infty)$.*

Theorem 3.2. *For all $n \in \mathbb{N}$ the zeros of $F(-n, -n \pm \frac{1}{3}; -2n; z)$ and $F(-n, -n \pm \frac{1}{3}; -2n \pm \frac{2}{3}; z)$ are all real and lie in the interval $(1, \infty)$.*

Proof. Since $-n-1 < -n-\frac{1}{3} < -n < -n+\frac{1}{3} < -n+1$, we have the result for $F(-n, -n \pm \frac{1}{3}; -2n; z)$ from Theorem 3.1. Both polynomials $F(-n, -n \pm \frac{1}{3}; -2n \pm \frac{2}{3}; z)$ are of the form $F(-n, b; 2b; z)$ with $b < 1-n$. Theorem 1.1(v) then yields the stated result. □

Corollary 3.3. (a) *The zeros of the polynomials $F(-n, -n \pm \frac{1}{3}; \frac{2}{3}; z)$ and $F(-n, n \pm \frac{1}{3}; \frac{4}{3}; z)$ are all real and negative.*

(b) *The zeros of the polynomials $F(-n, n+1; \frac{4}{3}; z)$, $F(-n, n+1; \frac{2}{3}; z)$, $F(-n, n+\frac{5}{3}; \frac{4}{3}; z)$ and $F(-n, n+\frac{1}{3}; \frac{2}{3}; z)$ are all real and lie in $(0, 1)$.*

Proof. (a) Applying the identity (2.1) to each of the polynomials given in Theorem 3.2 yields the result if we observe that $z > 1$ implies $1 - z < 0$.

(b) Applying the identity (2.2) to each polynomial in Theorem 3.2, the stated results follow. □

Theorem 3.4. *The zeros of $F(-n, n+1; 2n+2; z)$ all lie on the circle $|z-1| = 1$.*

Proof. The polynomial $F(-n, n+1; 2n+2; z)$ is of the form $F(-n, b; 2b; z)$ with $b > -\frac{1}{2}$ and the result follows from Theorem 1.1(i). □

Corollary 3.5. (a) *The zeros of $F(-n, n+1; -2n; z)$ all lie on the unit circle $|z| = 1$.*

(b) *The zeros of $F(-n, -3n-1; -2n; z)$ all lie on the straight line $Re(z) = \frac{1}{2}$.*

Proof. (a) This follows from Theorem 3.4 and identity (2.1).

(b) From (2.2) and Theorem 3.4 we deduce that the zeros of $F(-n, -3n-1; -2n; z)$ will all lie on the curve $|z-1| = |z|$ or $\mathrm{Re}(z) = \frac{1}{2}$. □

Theorem 3.6. *The zeros of $F(-n, (-n+1)/3; (-2n+2)/3; z)$ can be described as follows*:

(a) *If $n = 3k+1$, then $F(-n, (-n+1)/3; (-2n+2)/3; z) = F(-3k-1, -k; -2k; z)$ reduces to a polynomial of degree k and all its zeros lie on the straight line $Re(z) = \frac{1}{2}$.*

(b) *If* $n = 3k$, *then* $F(-n, (-n+1)/3; (-2n+2)/3; z)$ *has* k *zeros lying on the circle* $|z-1| = 1$. *The remaining* $2k$ *zeros are all non-real if* k *is even and if* k *is odd, there are* 2 *real zeros and* $(2k-2)$ *non-real zeros.*

(c) *If* $n = 3k+2$, *then* $F(-n, (-n+1)/3; (-2n+2)/3; z)$ *has* $(k+2)$ *zeros lying on the circle* $|z-1| = 1$. *The remaining* $2k$ *zeros are non-real if* k *is even while if* k *is odd,* 2 *are real and* $(2k-2)$ *zeros are non-real.*

Proof. (a) This follows directly from Corollary 3.5(b).

(b) The polynomial $F(-3k, -k+\frac{1}{3}; -2k+\frac{2}{3}; z)$ is of the form $F(-n, b; 2b; z)$ where $b = -k + \frac{1}{3}$. Since $b \in (-k-\frac{1}{2}, -k+\frac{1}{2})$, the result follows from Theorem 1.1(ii) if $n > 3$ and from Theorem 1.1(iii) if $n = 3$.

The proof of (c) follows the same arguments. $\square$

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